

# Linear Error Analysis and HVE

**Scott B. Anderson, Ph.D.**  
Anderson-Lee Applied Science

## Abstract

This monograph demonstrates the application of linear error analysis to the refinement of HVE simulations.

## What is Linear Error Analysis

Many non-linear dynamical systems will behave linearly with respect to small perturbations of their initial conditions. As a simple example, imagine trying to make a hole-in-one on a “putt-putt” golf course. The path of a struck golf ball will be a very non-linear function of the strength of the swing, the choice of the tee-off point, and the angle of the head of the putter. The ball may hit an obstacle or drop down a short-cut or bank off a corner depending on where the ball is teed up or how hard the putter is swung.

However, if an initial putt almost yields a hole-in-one, it is reasonable to expect that a very slight adjustment of the tee-off point or the strength of the swing could improve a second try at the hole. This is the essence of linear error analysis; to mathematically compute a better second try.

## The Mathematics

Examine a non-linear transformation from “initial” variables  $a$ ,  $b$ ,  $c$  (etc.) to “final” variables  $a'$ ,  $b'$ , and  $c'$ .

$$a' = f(a, b, c)$$

$$b' = g(a, b, c)$$

$$c' = h(a, b, c)$$

These relations can represent the result of an HVE run which transforms a vehicle in some initial state with some position, velocity, and orientation into a final state where the user often knows exactly what position, velocity, and orientation the vehicle had.

The usual goal for an HVE user is to determine the initial state given the final state. Suppose that by hook or crook or trial and error, a user makes a guess as to the initial state which leads to a reasonable (but not perfect) final state. We can represent this trial run as:

$$Q^{\text{Final Trial}} = F(Q^{\text{Initial Trial}})$$

and the goal of our analysis as:

$$Q^{\text{Final Actual}} = F(Q^{\text{Initial Actual}})$$

where we want to determine the actual initial  $Q$ . If the difference or “error”:

$$\Delta = Q^{\text{Final Actual}} - Q^{\text{Final Trial}} = F(Q^{\text{Initial Actual}}) - F(Q^{\text{Initial Trial}})$$

is small, it is reasonable to assume that the difference in initial conditions between trial and actual is small. This allows us to expand the actual initial conditions as:

$$Q^{\text{Initial Actual}} = Q^{\text{Initial Trial}} + \delta$$

Substituting this back into our relations and expanding yields:

$$F(Q^{\text{Initial Actual}}) = F(Q^{\text{Initial Trial}} + \delta) = F(Q^{\text{Initial Trial}}) + \partial F(Q^{\text{Initial Trial}}) \bullet \delta$$

where  $\partial F$  is the partial derivative of the non-linear function evaluated at the trial initial conditions. Further substitution has the result:

$$\Delta = \partial F(Q^{\text{Initial}}_{\text{Trial}}) \bullet \delta$$

Now if we have set things up so that  $\Delta$  consists of  $N$  differences and  $\delta$  is also  $N$ -dimensional, then  $\partial F$  is an  $N \times N$  dimensional matrix that can be inverted. Consequently, we can solve for the required change in initial conditions by evaluating:

$$\delta = \partial F^{-1} \bullet \Delta$$

We can now add these changes to the trial initial conditions to create a set of initial conditions that results in the actual final conditions. Unfortunately, it is rare that the exact desired final conditions are obtained, so one generally repeats the process, generating better and better initial conditions that reduce the error at each step.

## The Example

The following example illustrates how linear error analysis was actually applied to refine an HVE run. This example is taken from an actual collision that involved a semi and two cars. In this particular case, the front of a parked Buick was struck and the method of linear error analysis was used to refine the immediate post impact conditions. The initial position (X, Y) and orientation ( $\psi$ ) of the Buick

were reasonably known, but the initial speed (V), sideslip ( $\theta$ ), and angular velocity ( $\omega$ ) were not. After some experimentation, a promising set of initial conditions was found. This set is presented in the first column of Table 1 shown below. The known final resting position and orientation is presented in the second column from the right, labeled "Desired". These are the values we would like to arrive at from the simulation run. The actual values that were found after the HVE run are presented in the second column from the left, labeled "Final". The differences between the desired results and the actual run results are presented in the final column, labeled "Delta". It can be seen that the final Y position was not too far off from the known Y position but the orientation was off by  $10^\circ$  and the X position was off by almost 5 feet.

After that first run was completed, three additional "perturbing" runs were made in order to measure the sensitivity of the final position to changes in the initial conditions. The next run had the initial speed increased by 1 mph and the results were recorded in the third column. The run after that had the sideslip changed by  $5^\circ$  and the results recorded in the fourth column. The final run had the angular velocity changed by  $5^\circ/\text{s}$  with the results recorded in the fifth column.

Table 1: Linear error analysis first iteration results for The Example

| <b>Initial</b>                   | <b>Final</b>          | <b><math>\delta V = 1\text{mph}</math></b> | <b><math>\delta\theta = 5^\circ</math></b> | <b><math>\delta\omega = 5^\circ/\text{s}</math></b> | <b>Desired</b> | <b>Delta</b> |
|----------------------------------|-----------------------|--|--|---|----------------|--------------|
| V = 13.00 mph                    | X = 262.53 ft         | 264.04 ft                                  | 262.86 ft                                  | 262.71 ft   | 267.16 ft      | 4.63 ft      |
| $\theta = 160.00^\circ$          | Y = -12.45 ft         | -13.07 ft                                  | -11.01 ft                                  | -12.57 ft   | -13.40 ft      | -0.95 ft     |
| $\omega = 165.00^\circ/\text{s}$ | $\psi = 303.11^\circ$ | 310.28°                                    | 303.03°                                    | 306.20°   | 293.35°        | -9.76°       |

Figure 1 shows the initial trial state of the Buick. Figure 2 shows the desired final position of the Buick, while Figure 3 shows the actual final position of the Buick after the first trial.

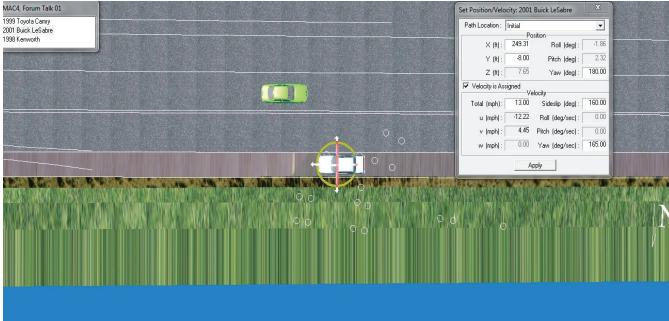


Figure 1: Initial trial state of the Buick

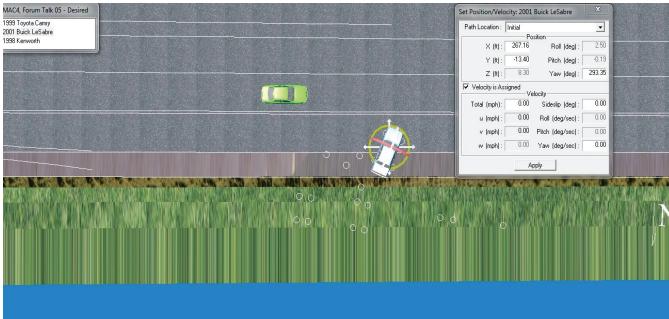


Figure 2: Desired final position of the Buick

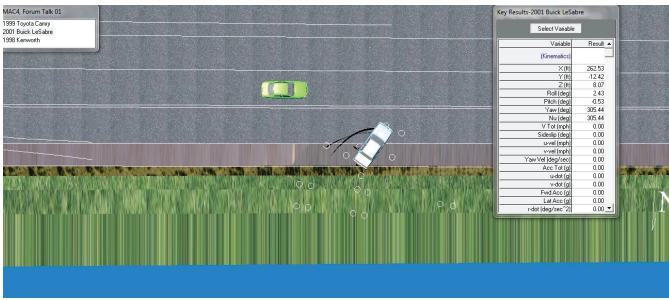


Figure 3: Actual final position of the Buick after the first run

As described, three runs were made by perturbing the initial trial position. Figure 4 illustrates the initial state of a perturbation of the initial speed by 1 mph, while Figure 5 illustrates the final result of perturbation of the initial sideslip angle by 5 degrees.



Figure 4: Initial state of a perturbation of the initial speed by 1 mph



Figure 5: Final result of a perturbation of the initial sideslip angle by 5 degrees

To compute the sensitivity, the results of the first run were subtracted from the results of the three subsequent perturbation runs. The computed differences produced a matrix of partial derivatives which was the “sensitivity” matrix:

| 1.51 ft/mph   | 0.066 ft° | 0.036 ft°/s  |
|---------------|-----------|--------------|
| -0.625 ft/mph | 0.288 ft° | -0.024 ft°/s |
| 7.17 °/mph    | -0.016 °° | 0.618 °°/s   |

The computation of the necessary adjustments to the initial conditions required the inverse of the sensitivity matrix. The inverse of this matrix could be computed and was found to be:

|               |               |              |
|---------------|---------------|--------------|
| 0.856 mph/ft  | -0.197 mph/ft | -0.058 mph/° |
| 1.029 °/ft    | 3.245 °/ft    | 0.067 °/°    |
| -9.880 °/s/ft | 2.389 °/s/ft  | 2.288 °/s/°  |

The adjustments to the initial conditions were found by taking this inverse matrix and multiplying it by the deltas displayed in the last column in the first table. The adjustments are:

|          |
|----------|
| 4.71 ft  |
| 1.028 ft |
| -70.34°  |

These adjustments were added to the values for the initial conditions in the first column of the first table and the process was repeated as shown in Table 2.

Notice that this second run was a bit closer in terms of distance from the desired final position. The initial trial run was 4.73 feet from the desired final position while this run was just 2.62 feet. Figure 6 displays the initial state of the Buick for the second iteration and Figure 7 shows the result of that second run.

This second run was still not close enough, so we iterated a third time. We computed the sensitivity matrix, then computed the inverse matrix, and finally the adjustments. Adding the adjustments to the initial conditions yielded Table 3.



Figure 6: Initial state of Buick for second iteration



Figure 7: Final state of Buick after second iteration

Notice in Table 3 that the trial initial conditions are starting to converge to better values. We were now only 0.29 feet from the desired position. In order to take advantage of this, the size of the perturbations of the initial conditions were scaled down, for example from 1 mph to 0.1 mph.

Table 2: Linear error analysis second iteration results for The Example

| Initial                         | Final                   | $\delta V = 1\text{mph}$ | $\delta\theta = 5^\circ$ | $\delta\omega = 5^\circ/\text{s}$ | Desired   | Delta    |
|---------------------------------|-------------------------|--------------------------|--------------------------|-----------------------------------|-----------|----------|
| $V = 17.71 \text{ mph}$         | $X = 269.46 \text{ ft}$ | 271.54 ft                | 270.09 ft                | 269.56 ft                         | 267.16 ft | -2.30 ft |
| $\theta = 161.02^\circ$         | $Y = -14.65 \text{ ft}$ | -15.29 ft                | -13.18 ft                | -14.56 ft                         | -13.40 ft | 1.25 ft  |
| $\omega = 94.66^\circ/\text{s}$ | $\psi = 274.53^\circ$   | 279.62°                  | 271.92°                  | 280.38°                           | 293.35°   | 18.82°   |

Table 3: Linear error analysis third iteration results for The Example

| Initial                          | Final                   | $\delta V = .1\text{mph}$ | $\delta\theta = 1^\circ$ | $\delta\omega = 1^\circ/\text{s}$ | Desired   | Delta    |
|----------------------------------|-------------------------|---------------------------|--------------------------|-----------------------------------|-----------|----------|
| $V = 16.39 \text{ mph}$          | $X = 267.20 \text{ ft}$ | 267.38 ft                 | 267.35 ft                | 267.20 ft                         | 267.16 ft | -0.04 ft |
| $\theta = 161.07^\circ$          | $Y = -13.69 \text{ ft}$ | -13.78 ft                 | -13.37 ft                | -13.72 ft                         | -13.40 ft | 0.29 ft  |
| $\omega = 116.46^\circ/\text{s}$ | $\psi = 290.41^\circ$   | 290.90°                   | 290.36°                  | 291.26°                           | 293.35°   | 2.94°    |

Figure 8 shows the initial state of the Buick for the third iteration. Figure 9 shows the final state of the Buick after the third iteration.



Figure 8: Initial state of Buick for third iteration



Figure 9: Final state of Buick after third iteration

We were still not as close as we would like to be with respect to the initial angular velocity. So we iterated a fourth time. We computed the sensitivity matrix, then computed the inverse matrix, and finally the adjustments one last time. Adding the adjustments to the initial conditions yielded the final table, Table 4 shown below. Notice that no

perturbed runs were made as the errors were now acceptable.

We found a set of initial conditions that yielded a much better final resting place for the Buick than our initial trial. We were only 0.17 feet away from the desired final position and only  $1^\circ$  from the desired yaw angle. This substantial improvement in the initial state of the run is the goal of linear error analysis.

## Discussion

In the preceding example, it took 13 HVE runs to finally generate the desired initial conditions. In addition, some computation of inverse matrices was required. In some cases, this time might exceed the time it would take an HVE user to arrive at these initial conditions by judicious experimentation. Consequently, there is a cost and benefit to linear error analysis. It is a method that provides an objective way to improve an estimate of the initial state but it is not a magic bullet to the general problem of finding that state.

One difficulty with using linear error analysis is that it assumes that the “region” of “phase space” about a run is linear with respect to small perturbations of the inputs. Somewhat surprisingly (or perhaps not), the results generated by HVE runs can be very non-linear with drastically different final states depending on the slightest change a user can make to the initial variables.

Table 4: Linear error analysis final iteration results for The Example

| Initial                          | Final                   | $\delta V = 0 \text{ mph}$ | $\delta \theta = 0^\circ$ | $\delta \omega = 0^\circ/\text{s}$ | Desired             | Delta              |
|----------------------------------|-------------------------|----------------------------|---------------------------|------------------------------------|---------------------|--------------------|
| $V = 16.26 \text{ mph}$          | $X = 267.06 \text{ ft}$ |                            |                           |                                    | $267.16 \text{ ft}$ | $0.10 \text{ ft}$  |
| $\theta = 162.34^\circ$          | $Y = -13.26 \text{ ft}$ |                            |                           |                                    | $-13.40 \text{ ft}$ | $-0.14 \text{ ft}$ |
| $\omega = 120.73^\circ/\text{s}$ | $\psi = 294.34^\circ$   |                            |                           |                                    | $293.35^\circ$      | $-0.99^\circ$      |

Consequently, some rough guidance can be offered for when a user may profitably use linear error analysis. For the motion of a single vehicle, sliding on a roadway, linear error analysis will probably help find the best initial conditions. For a somewhat simple collision between two vehicles (such as a T-bone type of collision), linear error analysis may help out. Complex collisions involving multiple vehicles are generally much too non-linear and linear error analysis is not recommended.

## **Author Contact Details**

Questions about the content of this paper can be directed to [scott@andlee.com](mailto:scott@andlee.com).